We here include some material complementary to the material covered from chapter 5 of Bondy and Murty.

**Proposition 5.1.** A graph is bipartite if and only if it does not contain an odd cycle.

**Proof.** If a graph is bipartite, we can paint the vertices using 2 colors, red and blue, so that no edge exists between two blue vertices or between two red vertices. If the graph contains an odd cycle, this is not possible.

To prove the reverse direction, assume the graph has no odd cycles, consider a spanning tree of the graph. Start from a vertex of this tree (call this vertex the root) and paint this vertex blue. Paint the children of the root on the tree red. Continuing along these lines, paint all the vertices of the tree either red or blue. Note that, if we were to only use the edges on the tree, the distance of the blue vertices from the root is even, while the distance of the red vertices is odd.

Now consider an edge $e$ of the graph that connects vertex $x$ to $y$. It is sufficient to show that (each) edge $e$ cannot connect two blue or two red vertices. If $e$ belongs in the spanning tree, then this is true from construction. Assume it does not belong in the tree. Then, there exists a cycle that consists of the edge $e$, and the path that connects vertex $x$ to vertex $y$ using only edges of the tree. This cycle from assumption is even. Thus, vertices $x$ and $y$ cannot have the same color.

**Some definitions**

A **matching** in a graph $G=(V,E)$ is a set $M \subseteq E$ of independent (pairwise distinct) edges. Recall that we can think of an edge as a set consisting of two vertices: the two vertices the edge is adjacent to. For example, in Fig 5.1, edge $AB$ can equivalently be thought as the set of vertices $\{A, B\}$. Two edges are called independent if they have no vertex in common. Similarly, two vertices are called independent if they do not have a common edge.

We will use the notation $\cup M = \cup_{e \in M} e$ to denote the set of matched vertices, that is, vertices that are adjacent to an edge in $M$. If $\cup M = V$ we say that $M$ is a **perfect matching**, i.e., all vertices are matched. That is, $M$ is a set of edges that “hit” all vertices. Let $v(G)$ denote the number of edges of the largest matching in $\Gamma$, that covers as many vertices as possible.

Related to matching is the notion of **vertex cover**. A vertex cover of $G$ is a set of vertices $W \subseteq V$ satisfying

$$\forall e \in E, W \cap e \neq 0.$$ 

That is, every edge in the graph is adjacent to at least one vertex in the cover. Note that if $W$ is a vertex cover, the remaining vertices in $V - W$ are independent. Let $\tau(G)$ denote the size of the smallest vertex cover of $G$. It is clear that

$$v(G) \leq \tau(G).$$ (5.1)
Indeed, the cover $\tau(G)$ includes at least one end from all edges in the graph, and thus from all edges in the maximum matching as well.

We cannot always satisfy (5.1) with equality. For example, consider an odd cycle with $2k + 1$ edges. Then $\tau(G) = k + 1$, while $v(G) = k$. In Figure 5.1, the set $W = \{A, B, D\}$ is a vertex cover, while edges $AB$ and $DE$ form a matching.

![Figure 5.1: An odd cycle with 2k+1=5 edges.](image)

Konigs theorem, states that equality holds in bipartite graphs. We will follow the proof in Chapter 5 of Bondy and Murty, but we restate here the theorem for completeness.

**Theorem 5.2.** Konigs theorem In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

### 5.0.1. An alternative proof of Halls theorem

We already saw in Chapter 5 of Bondy and Murty one proof of Halls theorem. Here we will give an alternative proof, that uses Konigs theorem.

**Theorem 5.3.** Halls theorem Let $G$ be a bipartite graph with the two parts $X$ and $Y$. Then $G$ contains a matching of $X$ if and only if

$$N(S) \geq S \text{ for all } S \subseteq X.$$ 

**Proof.** Assume that the marriage condition holds. We will prove a contradiction. If $G$ contains no matching of $X$, then from Konigs theorem there exists a cover $C$ that consists of less than $|X|$ vertices, say $C = A \cup B$ with $A \subset X$ and $B \subseteq Y$. Then

$$|A| + |B| = |C| < |X|$$  \hspace{1cm} (5.2)

and thus

$$|B| < |X| - |A| = |X - A|.$$  \hspace{1cm} (5.3)

Since $C$ is a cover, there do not exist any edges between $X - A$ and $Y - B$, thus all the neighbors of $X - A$ belong in $B$ and

$$N(X - A) \leq |B| \leq |X - A|.$$  

The marriage condition fails for the set of vertices $X - A$.\hfill \Box

Halls theorem has numerous applications, both outside graph theory and within. The following corollary gives a graph-theory application.

**Corollary 5.4.** Every regular graph of positive even degree has a 2-factor.

**Proof.** Let $G$ be a graph where all vertices have degree $2k$. Then we know that there exists an Euler cycle. We first give an orientation to the edges of the graph following this cycle. Thus each vertex will now have $k$ incoming and $k$ outgoing vertices.

We then replace each vertex $u$ with two vertices, a vertex $u^-$ that keeps all the incoming edges and a vertex $u^+$ that keeps all the outgoing edges. The resulting graph is now bipartite and $k$ regular, thus it has a 1-factor. Collapsing every vertex pair back into a single vertex we turn this 1-factor into a 2-factor.\hfill \Box
5.0.2. Stable Marriage Theorem

In many practical applications there exist some preferences regarding matchings. In particular, each vertex $u$ might have a preference regarding which of its adjacent edges wants to be used for the matching, an ordering $(\leq_u)$ of these edges. We call the orderings for all vertices in $G$ a set of preferences for $G$. We then call a matching $m$ stable if for every edge $e \in E - M$ there exists an edge $f \in M$ such that $e$ and $f$ have a common vertex $v$ with $e \leq_u f$. That is, $f$ is preferred to $e$.

Theorem 5.5. For every set of preferences, the bipartite graph $G$ has a stable matching.

Proof. Let $X$ and $Y$ be the two parts of the bipartite graph. We will say that matching $M_1$ is better than matching $M_2$ if $M_1$ makes the vertices in $Y$ happier than $M_2$ does. That is, every vertex $v \in Y$ matched to an edge $e$ in $M_2$, is also matched to an edge $f$ in $M_1$ with $e \leq_u f$. For a given matching $M$, a vertex $w \in X$ is said to be acceptable to vertex $u$ in $Y$ the edge $e = wu$ is not already used in the matching and for any edge $f \in M$ adjacent to $u$ it holds that $f \leq_u e$ (it replaces a less good matching). Also, a vertex $w \in X$ is happy with a matching $M$ if either it is unmatched, or its matching edge $e \in M$ satisfies $e > w$ for all edges $f = wv$ such that $w$ is acceptable to $v$.

We will now describe a greedy algorithm that finds a stable matching. Start with the empty matching and construct a sequence of matchings that each keep all the vertices in $X$ happy. Starting from such a matching, consider a vertex $w$ in $X$ that is unmatched but acceptable to some vertex $u$ in $Y$. Add to $M$ the maximal edge $wv$ such that $w$ is acceptable to $u$, and discard from $M$ any other edge at $u$.

Clearly, each matching in the sequence is better than the previous, and it is easy to check that they all keep the vertices in $X$ happy. So we continue until, at some point every unmatched vertex in $X$ is unacceptable to all its neighbors in $Y$. As every vertex in $X$ is happy with $M$, this matching is stable. \qed