Exercise 5.12

We derive the dual of the problem

\[
\begin{align*}
\text{minimize} & \quad - \sum_{i=1}^{m} \log y_i \\
\text{subject to} & \quad y = b - Ax,
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n} \) has \( a_i^T \) as its \( i \)th row. The Lagrangian is

\[
L(x, y, v) = - \sum_{i=1}^{m} \log y_i + v^T(y - b + Ax)
\]

and the dual function is

\[
g(v) = \inf_{x, y} \left( - \sum_{i=1}^{m} \log y_i + v^T(y - b + Ax) \right)
\]

The term \( v^T Ax \) is unbounded below as a function of \( x \) unless \( A^T v = 0 \). The terms in \( y \) are unbounded below if \( v \not\geq 0 \), and achieve their minimum for \( y_i = 1/v_i \), otherwise. We therefore find the dual function

\[
g(v) = \begin{cases} 
\sum_{i=1}^{m} \log v_i + m - b^T v & A^T v = 0, v \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

and the dual problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{m} \log v_i + m - b^T v \\
\text{subject to} & \quad A^T v = 0.
\end{align*}
\]

Exercise 5.13

(a) The Lagrangian is

\[
L(x, \mu, v) = c^T x + \mu^T (Ax - b) - v^T x + x^T \operatorname{diag}(v)x \\
= x^T \operatorname{diag}(v)x + (c + A^T \mu - v)^T x - b^T \mu.
\]

Minimizing over \( x \) gives the dual function

\[
g(\mu, v) = \begin{cases} 
- b^T \mu - (1/4) \sum_{i=1}^{n} (c_i + a_i^T \mu - v_i)^2 / v_i, & v \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]
where \( a_i \) is the \( i \)th column of \( A \), and we adopt the convention that \( a^2/0 = \infty \) if \( a \neq 0 \), and \( a^2/0 = 0 \) if \( a = 0 \). The resulting dual problem is

\[
\text{maximize} \quad -b^T \mu - (1/4) \sum_{i=1}^{n} (c_i + a_i^T \mu - v_i)^2/v_i \\
\text{subject to} \quad v \succeq 0.
\]

In order to simplify this dual, we optimize analytically over \( v \), by noting that

\[
\sup_{v_i \geq 0} \left( -\frac{1}{4} \frac{(c_i + a_i^T \mu - v_i)^2}{v_i} \right) = \begin{cases} 
(c_i + a_i^T \mu) & c_i + a_i^T \mu \leq 0 \\
0 & c_i + a_i^T \mu \geq 0
\end{cases}
= \min\{0, (c_i + a_i^T \mu)\}.
\]

This allows us to eliminate \( v \) from the dual problem, and simplify it as

\[
\text{maximize} \quad -b^T \mu + \sum_{i=1}^{n} \min\{0, c_i + a_i^T \mu\} \\
\text{subject to} \quad \mu \succeq 0.
\]

(b) We follow the hint. The Lagrangian and dual function of the LP relaxation are

\[
L(x, u, v, w) = c^T x + u^T (Ax - b) - v^T x + w^T (x - 1) \\
= (c + A^T u - v + w)^T x - b^T u - 1^T w
\]

\[
g(u, v, w) = \begin{cases} 
-b^T u - 1^T w & A^T u - v + w + c = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

The dual problem is

\[
\text{maximize} \quad -b^T - 1^T w \\
\text{subject to} \quad A^T u - v + w + c = 0. \quad u \succeq 0, v \succeq 0, w \succeq 0,
\]

which is equivalent to the Lagrange relaxation problem derived above. We conclude that the two relaxations give the same value.

**Exercise 5.20**

The Lagrangian is

\[
L(x, y, \lambda, v, z) = -c^T x + \sum_{i=1}^{m} y_i \log y_i - \lambda^T x + v(1^T x - 1) + z^T (P x - y) \\
= (-c - \lambda + v 1 + P^T z)^T x + \sum_{i=1}^{m} y_i \log y_i - z^T y - v.
\]

The minimum over \( x \) is bounded below if and only if

\[-c - \lambda + v 1 + P^T z = 0.\]
To minimize over $y$, we set the derivative with respect to $y_i$ equal to zero, which gives $\log y_i + 1 - z_i = 0$, and conclude that

$$\inf_{y_i \geq 0} (y_i \log y_i - z_i y_i) = -e^{z_i - 1}.$$ 

The dual function is

$$g(\lambda, v, z) = \begin{cases} 
- \sum_{i=1}^{m} e^{z_i - 1} - v - c - \lambda + v \mathbf{1} + P^T z = 0 \\
- \infty & \text{otherwise}
\end{cases}$$

The dual problem is

$$\begin{align*}
\text{maximize} & \quad - \sum_{i=1}^{m} e^{z_i - 1} - v \\
\text{subject to} & \quad P^T z - c + v \mathbf{1} \succeq 0.
\end{align*}$$

This can be simplified by introducing a variable $w = z + v \mathbf{1}$ (and using the fact that $1 = P^T \mathbf{1}$), which gives

$$\begin{align*}
\text{maximize} & \quad - \sum_{i=1}^{m} e^{w_i - 1} - v \\
\text{subject to} & \quad P^T w \succeq c.
\end{align*}$$

Finally we can easily maximize the objective function over $v$ by setting the derivative equal to zero (the optimal value is $v = -\log(\sum_i 1 - w_i)$), which leads to

$$\begin{align*}
\text{maximize} & \quad - \log(\sum_{i=1}^{m} e^{w_i}) - 1 \\
\text{subject to} & \quad P^T w \succeq c.
\end{align*}$$

This is a geometric program, in convex form, with linear inequality constraints (i.e., monomial inequality constraints in the associated geometric program).

**Exercise 5.26**

(a) The figure shows the feasible set (the intersection of the two shaded disks) and some contour lines of the objective function. There is only one feasible point, $(1, 0)$, so it is optimal for the primal problem, and we have $p^* = 1$.

(b) The KKT conditions are

$$(x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1,$$

$$\lambda_1 \geq 0, \lambda_2 \leq 0$$

$$2x_1 + 2\lambda_1 (x_1 - 1) + 2\lambda_2 (x_1 - 1) = 0$$

$$2x_2 + 2\lambda_1 (x_2 - 1) + 2\lambda_2 (x_2 + 1) = 0$$

$$\lambda_1 ((x_1 - 1)^2 + (x_2 - 1)^2 - 1) = \lambda_2 ((x_1 - 1)^2 + (x_2 + 1)^2 - 1) = 0.$$
At $x = (1, 0)$, these conditions reduce to
\[ \lambda_1 \geq 0, \lambda_2 \geq 0, 2 = 0, -2\lambda_1 + 2\lambda_2 = 0, \]
which (clearly, in view of the third equation) have no solution.

(c) The Lagrange dual function is given by
\[ g(\lambda_1, \lambda_2) = \inf_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2) \]
where
\[ L(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1((x_1 - 1)^2 + (x_2 - 1)^2 - 1) + \lambda_2((x_1 - 1)^2 + (x_2 + 1)^2 - 1) \]
\[ = (1 + \lambda_1 + \lambda_2)x_1^2 + (1 + \lambda_1 + \lambda_2)x_2^2 - 2(\lambda_1 + \lambda_2)x_1 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2. \]
$L$ reaches its minimum for
\[ x_1 = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2}, \quad x_2 = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2} \]
and we find
\[ g(\lambda_1, \lambda_2) = \begin{cases} \frac{-(\lambda_1 + \lambda_2)^2 + (\lambda_1 - \lambda_2)^2}{1 + \lambda_1 + \lambda_2} + \lambda_1 + \lambda_2 & 1 + \lambda_1 + \lambda_2 \geq 0 \\ -\infty & \text{otherwise,} \end{cases} \]
where we interpret $a/0 = 0$ if $a = 0$ and as $-\infty$ if $a < 0$. The Lagrange dual problem is given by
\[ \text{maximize} \quad (\lambda_1 + \lambda_2 - (\lambda_1 - \lambda_2)^2)/(1 + \lambda_1 + \lambda_2) \]
subject to $\lambda_1, \lambda_2 \geq 0$.

We see that $g(\lambda_1, \lambda_2)$ tends to 1 as $\lambda_1 \to \infty$. We have $d^* = p^* = 1$, but the dual optimum is not attained.
Recall that the KKT conditions only hold if (1) strong duality holds, (2) the primal optimum is attained, and (3) the dual optimum is attained. In this example, the KKT conditions fail because the dual optimum is not attained.