Exercise 2.2
Convex sets:

The intersection of two convex sets is convex. Therefore if \( S \) is a convex set, the intersection of \( S \) with a line is convex. Conversely, suppose the intersection of \( S \) with any line is convex. Take any two distinct points \( x_1, x_2 \in S \) and consider the line \( L \) through them. By hypothesis, \( S \cap L \) is convex and so any convex combination of \( x_1 \) and \( x_2 \) lies in both \( L \) and \( S \).

Affine sets:

Replace the word convex by affine in the above argument.

Exercise 2.4

Let \( H := \{ \sum \theta_i x_i : x_i \in S, \sum \theta_i = 1 \} \) be the convex hull of \( S \) and \( \mathcal{D} := \bigcap \{ D : D \text{ convex}, S \subset D \} \).

We first show \( H \subset \mathcal{D} \). Let \( x \) be a convex combination of points in \( S \), i.e. \( x \in H \), and let \( D \) be any convex set containing \( S \). \( D \) being convex, it contains any convex combination of points in \( S \), and so \( x \in D \). Hence \( H \subset \mathcal{D} \). We must now show \( \mathcal{D} \subset H \). But \( H \) is convex, hence it is by definition that \( \mathcal{D} \subset H \) since \( H \) is one of the sets \( D \) in the definition of \( \mathcal{D} \).

Exercise 2.5

The distance between the two hyperplanes is \( d = |b_1 - b_2|/\|a\|_2 \). Consider the line \( L \) through the origin in the direction of the normal vector \( a \). Let \( x_i = H_i \cap L \). \( L \) is perpendicular to both hyperplanes so \( d \) is the distance between \( x_1 \) and \( x_2 \). We have

\[
x_i = \frac{b_i}{\|a\|_2}a
\]

and this gives

\[
\|x_1 - x_2\| = \frac{|b_1 - b_2|}{\|a\|_2}
\]

Exercise 2.7

The norm being non-negative, we have \( \|x - a\| \leq \|x - b\| \) iff \( \|x - a\|^2 \leq \|x - b\|^2 \) and we get

\[
\|x - a\|^2 \leq \|x - b\|^2 \iff (x - a)^T(x - a) \leq (x - b)^T(x - b) \\
\iff x^T x - x^T a - a^T x + a^T a \leq x^T x - x^T b - b^T x + b^T b \\
\iff 2(b - a)^T x \leq b^T b - a^T a
\]
Taking \( c = (b - a) \) and \( d = b^T b - a^T a \), we see that this is indeed a halfspace. This makes good geometric sense: the points that are equidistant to \( a \) and \( b \) are given by a hyperplane whose normal is in the direction \( b - a \).

**Exercise 2.8**

a) Yes. If \( a_1 \) and \( a_2 \) are collinear, then the set is simply a segment, which is obviously a polyhedra. Let us therefore assume that \( a_1 \) and \( a_2 \) are independent. Let \( v_1, \ldots, v_{n-2} \) be \( n-2 \) linearly independent vectors orthogonal to the plane spanned by \( \{a_1, a_2\} \). Let \( c_1 \) be a vector in the plane spanned by \( a_1 \) and \( a_2 \) and orthogonal to \( a_2 \) and similarly let \( c_2 \) be a vector in the same plane and orthogonal to \( a_1 \). A point \( x \) in the plane spanned by \( a_1 \) and \( a_2 \) will be in \( S \) if and only if

\[
-|c_1^T a_1| \leq |c_1^T x| \leq |c_1^T a_1| \\
-|c_2^T a_2| \leq |c_2^T x| \leq |c_2^T a_2|
\]

We can therefore take as defining equations for \( S \)

\[
v_k^T x = 0 \quad \text{for} \quad k = 1, \ldots, n-2 \\
c_i^T x \leq |c_i a_i| \quad \text{for} \quad i = 1, 2 \\
c_i^T x \leq -|c_i a_i| \quad \text{for} \quad i = 1, 2
\]

b) Yes. It is clearly defined by 1 linear inequality and 3 equality constraints.

**Exercise 2.9**

a) This is quite similar to 2.7. Using the same reasoning, we get

\[
\|x - x_0\| \leq \|x - x_i\| \iff 2(x_i - x_0)^T x \leq x_i^T x_i - x_0^T x_0
\]

and so we get the result taking

\[
A = 2 \begin{pmatrix}
x_1 - x_0 \\
x_2 - x_0 \\
\vdots \\
x_K - x_0
\end{pmatrix} \quad b = \begin{pmatrix}
x_1^T x_1 - x_0^T x_0 \\
x_2^T x_2 - x_0^T x_0 \\
\vdots \\
x_K^T x_K - x_0^T x_0
\end{pmatrix}
\]

**Exercise 2.12**

a) This is the intersection of two halfspaces, hence it is a polyhedra and in particular convex.

b) Again, this is an intersection of a finite number of halfspaces, hence a polyhedra.

c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if \( b_1 = 0 \) and \( b_2 = 0 \).

d) This set is convex. It is an intersection of halfspaces

\[
\bigcap_{y \in S} \{x : \|x - x_0\| \leq \|x - y\|\}
\]
Observe that by exercise 2.9, for a fixed $y$, this set is a halfspace.

e) Not convex. Take $S = \{-1, 1\}$, $T = 0$. Then the set is not even connected!

**Exercise 2.21**

The conditions $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$, form a set of homogeneous linear inequalities in $(a, b)$. Therefore $K$ is the intersection of halfspaces that pass through the origin. Hence it is a convex cone. Note that this does not require convexity of $C$ or $D$. 